Continuum Plasticity

Largely from “Mechanical Metallurgy,”

Chapters 3 and 8
Stress-strain response

- Stress, \( s = \frac{P}{A_0} \)
- Strain, \( \varepsilon = \frac{\delta}{L_0} \)
- The \( s-\varepsilon \) response will depend on Temp., composition, strain rate, heat treatment, state of stress, history, etc.
- \( E, s_y, UTS, e_u, RA \)
Total Strain, \( \varepsilon = \varepsilon_{\text{elastic}} + \varepsilon_{\text{plastic}} \)

\( = \frac{\sigma}{E} + \varepsilon_{\text{plastic}} \)
True Stress-True Strain Curve

\[ \sigma = s(1+e) \]

\[ \varepsilon = \ln(1+e) \]

Also known as the flow curve.
Strain Hardening Exponent, $n$

$$\sigma = K \varepsilon^n$$

$K$ is known as strengthening coefficient.
## Work Hardening Data

<table>
<thead>
<tr>
<th>Material</th>
<th>Work Hardening Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stainless Steel</td>
<td>0.45-0.55</td>
</tr>
<tr>
<td>70/30 Brass</td>
<td>0.49</td>
</tr>
<tr>
<td>Copper (annealed)</td>
<td>0.3-0.54</td>
</tr>
<tr>
<td>Aluminum</td>
<td>0.15-0.25</td>
</tr>
<tr>
<td>Iron</td>
<td>0.05-0.15</td>
</tr>
<tr>
<td>0.05% C steel</td>
<td>0.26</td>
</tr>
</tbody>
</table>
Necking

• Volume of the specimen is constant during plastic deformation: \( AL = A_0 L_0 \)

• Initial strain hardening more than compensates for reduction in area. Engineering stress continues to raise with engineering strain.

• A point is reached where decrease in area > increase in strength due to strain hardening. Deformation gets localized. \( A \) decreases more rapidly than the load due strain hardening. Further elongation occurs with decreasing load.
Considère’s Criterion

“Necking begins when the increase in stress due to decrease in the cross-sectional area is greater than the increase in load bearing capacity of the specimen due to work hardening.”

\[ dP = 0 = \sigma dA + A d\sigma \Rightarrow \frac{d\sigma}{\sigma} = -\frac{dA}{A} \]

Volume preservation \( \Rightarrow -\frac{dA}{A} = \frac{dL}{L} = d\varepsilon \)

Combining, \( \frac{d\sigma}{d\varepsilon} = \sigma \). Necking begins at a point where rate of strain hardening is equal to the stress.

In terms of engineering values, \( \frac{ds}{de} = 0 \), at max. s!!
Figure 3-2 Idealized flow curves. (a) Rigid ideal plastic material; (b) ideal plastic elastic region; (c) piecewise linear (strain-hardening) material.
Yield Criteria

In uniaxial loading, plastic flow begins when $\sigma = \sigma_0$, the tensile yield stress.

When does yielding begin when a material is subjected to an arbitrary state stress?
Yield Criteria for Metals

• Pure hydrostatic pressure or mean stress tensor, $\sigma_m$, doesn’t cause yielding in metals.

• Only the deviatoric stress, $\sigma'_{ij}$, which represents the shear stresses causes plastic flow.

• For an isotropic solid, the yield criterion must be independent of the choice of the axes, i.e., it must be an invariant function.

∴ Yield criterion must be some function of the J’s.
Von Mises’ Yield Criterion

- “Yielding would occur when $J_2$ exceeds some critical value” \( J_2=k^2 \)
- Yielding in uniaxial tension: \( \sigma_1=\sigma_0, \sigma_2=\sigma_3=0 \)
  \[
  J_2 = \frac{1}{6} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]
  \]
  \[
  \sigma_0/\sqrt{3} = k
  \]

\[
\sigma_0 = \frac{1}{\sqrt{2}} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2}
\]

\[
\sigma_0 = \frac{1}{\sqrt{2}} \left[ (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2) \right]^{1/2}
\]
• Pure shear (torsion test): $\sigma_1=-\sigma_3=\tau$, $\sigma_2=0$

• At yielding:

$$\sigma_1^2 + \sigma_1^2 + 4\sigma_1^2 = 6k^2$$

$\therefore \tau=k$

• Yield stress in pure tension is higher!

$$\sigma_0/ \tau = \sqrt{3}$$
Energy Equivalence

• Hencky (1924) showed that Von Mises yield criterion is equivalent to assuming “yielding occurs when the distortion energy reaches a critical value.”

• Elastic strain energy per unit volume, $U_0$

$$U_0 = \frac{1}{2E} (\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\nu}{E} (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x)$$

$$+ \frac{1}{2G} (\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)$$
In terms of principal stresses:

\[ U_0 = \frac{1}{2E} \left[ \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_3\sigma_2 + \sigma_1\sigma_3) \right] \]

In terms of invariants of the stress tensors:

\[ U_0 = \frac{1}{2E} \left[ I_1^2 - 2I_2(1 + \nu) \right] \]

Expressing in terms of bulk modulus (K) representing vol. change and shear modulus (G) representing distortion

\[ E = \frac{9GK}{3K + G} \quad \nu = \frac{3K - 2G}{6K + 2G} \]
\[ U_0 = \frac{I_1^2}{18K} + \frac{1}{6G} (I_1^2 - 3I_2) \]

- The first term on the RHS is dependent on change in volume and the second on distortion.

\[ U_{0,\text{dist}} = \frac{1}{12G} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \]

- For a uniaxial state of stress,

\[ U_{0,\text{dist}} = \frac{1}{12G} 2\sigma_0^2 \]
Tresca (Max. Shear Stress) Criterion

“Yielding occurs when the max. shear stress reaches the value of shear yield stress in the uniaxial tension test.”

- Max shear stress, \( \tau_{\text{max}} = (\sigma_1 - \sigma_3)/2 \)
- In uniaxial tension, \( \tau_0 = \sigma_0 / 2 \)
- Tresca criterion: \((\sigma_1 - \sigma_3) = \sigma_0\)
- Pure shear: \(\sigma_1 = -\sigma_3 = k; \sigma_2 = 0\)
  \[(\sigma_1 - \sigma_3) = 2k = \sigma_0 \Rightarrow k = \sigma_0 / 2\]
- Predicts the same stress for yielding in uniaxial tension and in pure shear
• Less complicated mathematically than von Mises criterion
• Often used in engineering design
• Doesn’t consider $\sigma_2$
• Need to know *apriori* the max. and min. principal stresses
• General form:

$$4J_2^3 - 27J_3^2 - 36k^2 J_2^2 + 96k^4 J_2 - 64k^6 = 0$$
Yield Locus

\[ \sigma_0 = \frac{1}{\sqrt{2}} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2} \]

For a biaxial plane-stress condition (\(\sigma_2 = 0\)); the von Mises criterion can be expressed as

\[ \sigma_1^2 + \sigma_3^2 - \sigma_1 \sigma_3 = \sigma_0^2 \]

Equation of an ellipse whose major semi-axis is \(\sqrt{2}\sigma_0\) and minor semi-axis is \(\sqrt{(2/3)}\sigma_0\)
Von Mises and Tresca predict the same yield stress for uniaxial and balanced biaxial stress loading. Max. difference (15.5%) for pure shear case.
Anisotropy in Yielding

Experiment

Isotropy

Stress in GPa

$\epsilon = 0.002$

Ti – 4Al – $\frac{1}{4}O_2$
Yield Surface

The yield criteria can be represented geometrically by a cylinder oriented at equal angles to the $\sigma_1$, $\sigma_2$, & $\sigma_3$ axes.

- A state of stress which gives a point inside the cylinder represents elastic behavior.
- Yielding begins when the state of stress reaches the surface of the cylinder.
- $MN$, the cylinder radius is the deviatoric stress.
• The cylinder axis, OM, which makes equal angles with the principal axes represents the hydrostatic component of the stress tensor.
• The generator of the yield surface is the line parallel to OM. If stress state characterized by \((\sigma_1, \sigma_2, \sigma_3)\) lies on the yield surface, so does \((\sigma_1+H, \sigma_2+H, \sigma_3+H)\)
• Von Mises criterion is represented by a right circular cylinder whereas the Tresca criterion is represented by a regular hexagonal prism.
Normality

- Drucker (1951): “The total plastic strain vector, must be normal to the yield surface.”
- Net work has to be expended during the plastic deformation of a body. So the rate of energy dissipation is nonnegative:
  \[ \sigma_{ij} d\varepsilon_{ij}^p \geq 0 \]
- The \( d\varepsilon_{ij}^p \) is the incremental plastic strain vector and must be normal to the yield surface.
- Because of the normality rule, the yield locus is always convex.
Hardening Models

• How does the yield surface change during plastic deformation?

• *Isotropic Hardening*: The yield surface expands uniformly, but with a fixed shape (e.g. ellipse in the case of von Mises solid) and fixed center.
Bauschinger Effect

The lowering of yield stress when deformation in one direction is followed by deformation in opposite direction.

Bauschinger 1881
Kinematic Hardening

The yield surface does not change its shape and size, but simply translates in the stress space in the direction of its normal.

Accounts for the Bauschinger effect.
Invariants of Stress and Strain

• Useful to simplify the representation of a complex state of stress or strain by means of respective invariant functions in such a way that the flow curve is unaltered.

• Most frequently used invariant functions:
  Effective stress, $\bar{\sigma}$, and effective strain, $\bar{\varepsilon}$

\[
\bar{\sigma} = \frac{1}{\sqrt{2}} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2}
\]

\[
d\bar{\varepsilon} = \frac{\sqrt{2}}{3} \left[ (d\varepsilon_1 - d\varepsilon_2)^2 + (d\varepsilon_2 - d\varepsilon_3)^2 + (d\varepsilon_3 - d\varepsilon_1)^2 \right]^{1/2}
\]
\[
\bar{\sigma} = \sqrt{3J_2} = \sqrt{\frac{3}{2}} S_{ij} S_{ij} \quad S_{ij} = \sigma'_{ij}
\]

\[
d\bar{\varepsilon} = \sqrt{\frac{2}{3}} d\varepsilon^P_{ij} d\varepsilon^P_{ij} \quad \varepsilon^\text{Plastic}_{ij} = \varepsilon^\text{Total}_{ij} - \varepsilon^\text{Elastic}_{ij}
\]

Numerical constants are chosen such that, in Uniaxial tension,

\[
d\bar{\varepsilon} = d\varepsilon_1 = -2d\varepsilon_2 = -2d\varepsilon_3
\]

Power law hardening:

\[
\bar{\sigma} = K\bar{\varepsilon}^n
\]
Plastic Stress-Strain Relations

• In elastic regime, the stress-strain relations are uniquely determined by the Hooke’s law.
• In plastic deformation, the strains also depend on the history of loading.

\[ \therefore \text{It is necessary to determine the differentials or increments of plastic strains throughout the loading path and then obtain the total strain by integration.} \]
Example

- A rod, 50 mm long, is extended to 60 mm and then compressed back to 50 mm.
- On the basis of total deformation:

\[ \varepsilon = \int_{50}^{60} \frac{dL}{L} + \int_{60}^{50} \frac{dL}{L} = 0 \]

On an incremental basis:

\[ \varepsilon = \int_{50}^{60} \frac{dL}{L} + \int_{60}^{50} - \frac{dL}{L} = 2 \ln 1.2 = 0.365 \]
Proportional Loading

A particular case in which all the stresses increase in the same ratio, i.e.,

\[
\frac{d\sigma_1}{\sigma_1} = \frac{d\sigma_2}{\sigma_2} = \frac{d\sigma_3}{\sigma_3}
\]

Plastic strains are independent of the loading path.
Two general categories of plastic stress-strain relationships.

- *Incremental or flow theories* relate stresses to plastic strain *increments*.
- *Deformation or total strain theories* relate the stresses to *total* plastic strains. Simpler mathematically.

- Both are the same for proportional loading!
Levy-Mises Equations

• Ideal plastic solids where elastic strains are negligible.

• Consider yielding under uniaxial tension:

$$\sigma_1 \neq 0, \sigma_2 = \sigma_3 = 0, \text{and } \sigma_m = \sigma_1 / 3$$

• Since only deviatoric stresses cause yielding

$$\sigma_1' = \sigma_1 - \sigma_m = \frac{2\sigma_1}{3}; \sigma_2' = \sigma_3' = \frac{-\sigma_1}{3}$$

• Constant vol. condition:

$$d\varepsilon_1 = -2d\varepsilon_2 = -2d\varepsilon_3$$

$$\frac{d\varepsilon_1}{d\varepsilon_2} = -2 = \frac{\sigma_1'}{\sigma_2'}$$
• Generalization: \[
\frac{d\varepsilon_1'}{\sigma_1'} = \frac{d\varepsilon_2'}{\sigma_2'} = \frac{d\varepsilon_3'}{\sigma_3'} = d\lambda
\]

• At any instant of deformation, the ratio of the plastic strain increments to the current deviatoric stresses is constant.

• In terms of actual stresses,

\[
d\varepsilon_1 = \frac{2}{3} d\lambda \left[ \sigma_1 - \frac{1}{2} (\sigma_2 + \sigma_3) \right]
\]

etc.

Recall, \[
\sigma_{ij} = \sigma_{ij}' + \frac{1}{3} \delta_{ij} \sigma_{kk}
\]
• Using the effective strain concept to evaluate $\lambda$, which yields

$$\overline{\sigma} = |\sigma_1|$$

$$d\overline{\varepsilon} = d\varepsilon_1 = -2d\varepsilon_2 = -2d\varepsilon_3$$

$$d\varepsilon_{ij} = \frac{3}{2} \frac{d\overline{\varepsilon}}{\overline{\sigma}} \sigma'_{ij} \iff$$

$$\varepsilon_x = \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$d\varepsilon_1 = \frac{d\overline{\varepsilon}}{\overline{\sigma}} \left[ \sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3) \right]$$

$$d\varepsilon_2 = \frac{d\overline{\varepsilon}}{\overline{\sigma}} \left[ \sigma_2 - \frac{1}{2}(\sigma_1 + \sigma_3) \right]$$

$$d\varepsilon_3 = \frac{d\overline{\varepsilon}}{\overline{\sigma}} \left[ \sigma_3 - \frac{1}{2}(\sigma_2 + \sigma_1) \right]$$

$$d\varepsilon = \frac{2}{3} d\lambda \overline{\sigma}$$
Drawback: Only plastic strains are considered.

Evaluation

\[
\begin{align*}
\frac{d\varepsilon_1}{\sigma} &= \frac{d\varepsilon}{\sigma} \left[ \sigma_1 - \frac{1}{2} (\sigma_2 + \sigma_3) \right] \\
\frac{d\varepsilon_2}{\sigma} &= \frac{d\varepsilon}{\sigma} \left[ \sigma_2 - \frac{1}{2} (\sigma_1 + \sigma_3) \right] \\
\frac{d\varepsilon_3}{\sigma} &= \frac{d\varepsilon}{\sigma} \left[ \sigma_3 - \frac{1}{2} (\sigma_2 + \sigma_1) \right]
\end{align*}
\]

\[
\cot \theta = \frac{d\varepsilon}{\sigma}
\]
Prandtl-Reuss Equations

- Proposed by Prandtl (1925) and Ruess (1930) for Elastic-Plastic Solid
- Considers elastic strains as well

\[ d\varepsilon_{ij}^T = d\varepsilon_{ij}^E + d\varepsilon_{ij}^P \]

Recall,

\[ \varepsilon_{ij}^E = \frac{1 + \nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij} \]

\[ \therefore \quad d\varepsilon_{ij}^E = \frac{1 + \nu}{E} d\sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij} \]

\[ = \frac{1 + \nu}{E} d\sigma_{ij}' + \frac{1 - 2\nu}{E} \frac{d\sigma_{kk}}{3} \delta_{ij} \]
• Combining with the plastic strain increments (given by Levy -Mises Equation)

\[ d\varepsilon_{ij}^T = \frac{1 + \nu}{E} d\sigma'_{ij} + \frac{1 + 2\nu}{E} d\sigma_{kk} \delta_{ij} + \frac{3}{2} \frac{d\bar{\varepsilon}}{\bar{\sigma}} \sigma'_{ij} \]

• Use these with a yield criterion and the equation that characterizes the flow behavior of the material (e.g. power-law hardening) to calculate the strain increment for an increment in load.

• The complete solution must also satisfy equilibrium, \( \varepsilon-\delta \) relations, and the BCs.
Summary

General Theory of Plasticity requires the following

1) A **yield criterion**, which specifies the onset of plastic deformation for different combinations of applied load. e.g. von Mises and Tresca

2) A **hardening rule**, which prescribes the work hardening of the material and the change in yield condition with the progression of plastic deformation.
   Isotropic or kinematic: power-law hardening

3) A **flow rule** which relates increments of plastic deformation to the stress components.
   e.g. Levy-Mises or Prandtal-Reuss